Abelian logic and the logics of pointed lattice-ordered varieties

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Abstract

We consider the class of pointed varieties of algebras having a lattice term reduct and we show that each such variety gives rise in a natural way, and according to a regular pattern, to at least three interesting logics. Although the mentioned class includes several logically and algebraically significant examples (e.g. Boolean algebras, MV algebras, Boolean algebras with operators, residuated lattices and their subvarieties, algebras from quantum logic or from depth relevant logic), we consider here in greater detail Abelian ℓ-groups, where such logics respectively correspond to: i) Meyer and Slaney's *Abelian logic* [26]; ii) Galli et al.'s *logic of equilibrium* [18]; iii) a new logic of "preservation of truth degrees".

1 Introduction

Abelian logic, the logic of Abelian ℓ-groups, was independently introduced by Meyer and Slaney [26] and by Casari [6] in the 1980's. Since then, a number of papers (e.g. [27], [29], [28], [23], [24]) have been occasionally devoted to the topic[1], which however never really made its way into the mainstream of logical research. Back when the logic was introduced, the paradigm of *abstract algebraic logic* (AAL: cp. e.g. [7], [13]) was not yet dominant in the investigations into the relation between logics and the corresponding classes of algebras; furthermore, it was still commonplace to characterise propositional logics mainly as collections of *theorems*, rather than by means of consequence relations. This was, in fact, the point of view that Meyer and Slaney adopted most of the time in their paper (unlike Casari, whose article does not have Abelian logic, however, as its main focus).

In this paper, we aim at departing from the above-mentioned trends and at reconsidering Abelian logic in a different perspective. We will place ourselves in the wider context of a study of pointed varieties of algebras with a

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[1] More or less extensive asides on Abelian logic are also contained in the books [31], [30], [20].

lattice term reduct - a class of varieties which includes several logically and algebraically significant examples (e.g. Boolean algebras, MV algebras, Boolean algebras with operators, residuated lattices and their subvarieties, algebras from quantum logic or from depth relevant logic) - and we will show that, given a rather minimal set of assumptions, each such variety gives rise in a natural way, and according to a regular pattern, to at least three interesting logics, two of which are finitely and strongly algebraisable, while the third need not even be protoalgebraic. In the case of Abelian ℓ-groups, such logics respectively correspond to: i) the consequence relation associated by Meyer and Slaney to Abelian logic; ii) Galli et al.'s *logic of equilibrium* [18]; iii) a new non-protoalgebraic but selfextensional logic of "preservation of truth degrees" (in the sense of [19], [11]).

Studying Abelian logic with the tools of AAL, and in this wider perspective, ties up very nicely with the *reverse algebraisation* programme (e.g. [4], [10], [32]), a successful research stream whose goal is extracting interesting deductive systems from classes of algebras which are significant in their own right. A thorough exploration of the logics arising from arbitrary pointed lattice-ordered varieties, however, must be left for another occasion: throughout this paper, we will mainly emphasise this specific application rather than trying to investigate to a greater depth the general case.

We close this introduction with some notational preliminaries. We will consider sentential logics formulated in languages with a denumerable stock of propositional variables $p_0, p_1 ...$ and finitely many connectives, including the nullary connective e. We assume that, among the defined connectives of every such language $\mathcal{L}$, there are at least two binary connectives $\wedge$ and $\vee$. We follow the convention according to which (primitive or defined) unary connectives bind stronger than binary connectives; moreover, we assume that $\wedge, \vee$ bind stronger than other (primitive or defined) binary connectives. In the special case where we deal with the language $\mathcal{L}^{\mathbb{A}}$ of Abelian logic, the complete list of primitive connectives includes: e (nullary), $-$ (unary), $\wedge, \vee, +$ (binary). However, we will often trade in e for the more familiar symbol 0.

As it is customary to do in AAL, we will not distinguish between logical languages and algebraic similarity types; so, $\mathtt{Fm}(\mathcal{L})$ will denote both the set of all formulas of $\mathcal{L}$ (seen as a logical language) and the set of all terms of $\mathcal{L}$ (seen as an algebraic similarity type). The letters α, β are used as metavariables for arbitrary members of $\mathtt{Fm}(\mathcal{L})$. By writing $\alpha\left[p_1, ..., p_n\right]$ we mean that the set of propositional variables occurring in α is a (proper or improper) subset of $\{p_1, ...p_n\}$, while by $\alpha\left[\beta_1/\alpha_1, ..., \beta_n/\alpha_n\right]$ we indicate the result of replacing one or more occurrences of β_i in α by α_i. If $\mathbf{A}$ is an algebra of type $\mathcal{L}$ and $a_1, ..., a_n \in A$, by $\alpha^{\mathbf{A}}\left(a_1, ..., a_n\right)$ we denote the result of the application of the term operation $\alpha^{\mathbf{A}}$ to the elements $a_1, ..., a_n$. We will feel free to employ the self-explanatory vectorial notations $\alpha\left[\overrightarrow{p}\right]$ and $\alpha^{\mathbf{A}}\left(\overrightarrow{a}\right)$ whenever the lengths of the indicated strings are clear from the context.

$\mathbb{A}$ will denote the variety of Abelian ℓ-groups, and $\mathbf{Z} = \langle Z, +, -, 0, \wedge, \vee \rangle$ the ℓ-group of the integers. For basic definitions and result concerning ℓ-groups, the reader is referred e.g. to [8].

2 Logics of pointed ℓ-varieties and Abelian logic

Definition 1 *A variety* $\mathbb{V}$ *is called a* pointed ℓ-variety (w.r.t. e)*, or for short an* e-ℓ-variety*, if the following two conditions are satisfied:*

- *the similarity type* $\mathcal{L}$ *of* $\mathbb{V}$ *is finite and includes a constant* e*;*
- *each* $\mathbf{A} \in \mathbb{V}$ *has a term reduct* $\left\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}} \right\rangle$ *which is a lattice.*

If $\mathbb{V}$ is a pointed ℓ-variety and $\mathbf{A} \in \mathbb{V}$, the *positive cone* $P^{\mathbf{A}}$ of $\mathbf{A}$ is the set $\left\{ a \in A : e^{\mathbf{A}} \leq^{\mathbf{A}} a \right\}$.

Definition 2 *Let* $\mathbb{V}$ *be a pointed* ℓ*-variety of type* $\mathcal{L}$*, and let* $\Gamma \cup \{\alpha\}$ *be a finite subset of* $\mathtt{Fm}(\mathcal{L})$*. We introduce the following logics out of* $\mathbb{V}$*:*

$$\Gamma \vDash_1^{\mathbb{V}} \alpha \qquad \text{iff for any } \overrightarrow{a} \in \mathbf{A} \in \mathbb{V},\ \left\{ \gamma^{\mathbf{A}}\left(\overrightarrow{a}\right) : \gamma \in \Gamma \right\} \subseteq \left\{ e^{\mathbf{A}} \right\} \text{ implies } \alpha^{\mathbf{A}}\left(\overrightarrow{a}\right) = e^{\mathbf{A}};$$

$$\Gamma \vDash_2^{\mathbb{V}} \alpha \qquad \text{iff for any } \overrightarrow{a} \in \mathbf{A} \in \mathbb{V},\ \left\{ \gamma^{\mathbf{A}}\left(\overrightarrow{a}\right) : \gamma \in \Gamma \right\} \subseteq P^{\mathbf{A}} \text{ implies } \alpha^{\mathbf{A}}\left(\overrightarrow{a}\right) \in P^{\mathbf{A}};$$

$$\Gamma \vDash_3^{\mathbb{V}} \alpha \qquad \text{iff for any } \overrightarrow{a} \in \mathbf{A} \in \mathbb{V},\ \bigwedge^{\mathbf{A}} \left\{ \gamma^{\mathbf{A}}\left(\overrightarrow{a}\right) : \gamma \in \Gamma \right\} \leq^{\mathbf{A}} \alpha^{\mathbf{A}}\left(\overrightarrow{a}\right).$$

It is clear that $\Gamma \vDash_1^{\mathbb{V}} \alpha$ iff $\{\gamma \approx e : \gamma \in \Gamma\} \vDash_{\mathtt{Eq}(\mathbb{V})} \alpha \approx e$, which means that $\vDash_1^{\mathbb{V}}$ is nothing but the e-assertional logic of $\mathbb{V}$. In the terminology of [5], on the other hand, $\vDash_2^{\mathbb{V}}$ is the $\langle e, e \wedge a \rangle$-assertional logic of $\mathbb{V}$. We now provide a few examples by way of illustration.

Example 3 *The variety* $\mathbb{BA}$ *of Boolean algebras is a* $\top$*-*ℓ*-variety, and* $\vDash_1^{\mathbb{BA}} = \vDash_2^{\mathbb{BA}} = \vDash_3^{\mathbb{BA}}$ *is nothing but classical propositional logic. The same holds for Heyting algebras and intuitionistic logic.*

Example 4 *The variety* $\mathbb{MV}$ *of MV algebras is a* 1*-*ℓ*-variety, and* $\vDash_1^{\mathbb{MV}} = \vDash_2^{\mathbb{MV}}$ *is Łukasiewicz infinite-valued logic, while* $\vDash_3^{\mathbb{MV}}$ *is the "Łukasiewicz logic that preserves degrees of truth" investigated in [11].*

Example 5 *The variety* $\mathbb{NMA}$ *of normal modal algebras is a* $\top$*-*ℓ*-variety, and* $\vDash_1^{\mathbb{NMA}} = \vDash_2^{\mathbb{NMA}}$ *is the global (or strong) consequence relation of the modal logic* $\mathbf{K}$*. On the other hand,* $\vDash_3^{\mathbb{NMA}}$ *is its local (or weak) necessitation-free counterpart.*

As illustrated by the preceding examples, whenever e realises the top element of the lattice reduct in all algebras of $\mathbb{V}$, the logics $\vDash_1^{\mathbb{V}}$ and $\vDash_2^{\mathbb{V}}$ necessarily concide. In the next examples, all of the three logics are distinct from one another.

Example 6 *The variety* $\mathbb{FL}$ *(resp.* $\mathbb{RL}$*) of* $\mathbf{FL}$*-algebras (resp. residuated lattices) [16] is a* e*-*ℓ*-variety, and* $\vDash_2^{\mathbb{FL}}$ *(resp.* $\vDash_2^{\mathbb{RL}}$*) is (the positive fragment of) the substructural logic* $\mathbf{FL}$*, i.e. intuitionistic noncommutative linear logic without exponentials and without additive constants. The* e*-assertional logic of* $\mathbb{FL}$ $\vDash_1^{\mathbb{FL}}$ *has been briefly considered in [16], while the logic* $\vDash_3^{\mathbb{FL}}$ *is currently under investigation for the commutative, integral case (see [14]).*

Example 7 *The variety $\mathbb{A}$ of Abelian ℓ-groups is a 0-ℓ-variety. It follows from results in [18] that the logic $\vDash_1^{\mathbb{A}}$is definitionally equivalent to Galli et al.'s "logic of equilibrium" (being the 0-assertional logic of Abelian ℓ-groups). The logic $\vDash_2^{\mathbb{A}}$is the consequence relation that Meyer and Slaney had in mind when introducing their Abelian logic. Finally, the logic $\vDash_3^{\mathbb{A}}$is the analogue for Abelian ℓ-groups of the logic of preservation of truth degrees of [11].*

In some cases, the above logics can be equivalently defined in terms of homomorphisms into a single privileged algebra, rather than in terms of homomorphisms into arbitrary algebras in the variety. In the case of Abelian ℓ-groups, for example, we can confine ourselves to homomorphisms into $\mathbf{Z}$, for the integers generate Abelian ℓ-groups as a quasivariety [22].

So far we have been rather pedantic in our policy concerning subscripts and superscripts. Hereafter, they will be dropped whenever convenient and whenever no danger of ambiguity is impending, for the sake of notational simplicity.

The inclusion relationships between such logics are illustrated in the next Lemma.

Lemma 8 *(i) $\vDash_3^{\mathbb{V}}\subseteq\vDash_2^{\mathbb{V}}$; (ii) No other inclusion among the three logics need hold. In particular, if $\mathbb{V}=\mathbb{A}$, the preceding inclusion is proper and no other inclusion holds.*

Proof. (i) Suppose $\Gamma\vDash_3^{\mathbb{V}}\alpha$, and let $\overrightarrow{a}\in A\in\mathbb{V}$ be such that $\gamma(\overrightarrow{a})\geq e$ for every $\gamma\in\Gamma$. Then $\bigwedge\{\gamma(\overrightarrow{a}):\gamma\in\Gamma\}\geq e$ by lattice properties, whence $\alpha(\overrightarrow{a})\geq e$.

(ii) In $\mathbb{A}$ the preceding inclusion is proper for $\alpha+\alpha\vDash_2^{\mathbb{A}}\alpha$, yet $\alpha+\alpha\nvDash_3^{\mathbb{A}}\alpha$. $\vDash_1^{\mathbb{A}}$ is uncomparable with $\vDash_3^{\mathbb{A}}$: $\alpha+\alpha\nvDash_3^{\mathbb{A}}\alpha$ but $\alpha+\alpha\vDash_1^{\mathbb{A}}\alpha$, while conversely $\alpha\wedge\beta\vDash_3^{\mathbb{A}}\alpha$ yet $\alpha\wedge\beta\nvDash_1^{\mathbb{A}}\alpha$. $\vDash_1^{\mathbb{A}}$ is uncomparable with $\vDash_2^{\mathbb{A}}$: $-\alpha\nvDash_2^{\mathbb{A}}\alpha$ but $-\alpha\vDash_1^{\mathbb{A}}\alpha$, while conversely $\alpha\wedge\beta\vDash_2^{\mathbb{A}}\alpha$ yet $\alpha\wedge\beta\nvDash_1^{\mathbb{A}}\alpha$. ■

In general, the logic $\vDash_3^{\mathbb{V}}$ need not have theorems: it does if and only if $\mathbb{V}$ is a variety of *bounded* algebras where the top element is term definable.

Lemma 9 *$\vDash_3^{\mathbb{V}}$ has theorems iff every $\mathbf{A}\in\mathbb{V}$ has a uniformly term definable top element (w.r.t. the underlying lattice order).*

Proof. Suppose $\vDash_3^{\mathbb{V}}\alpha$. Then, for every propositional variable p not occurring in α, $p\vDash_3^{\mathbb{V}}\alpha$, which means that for every $\overrightarrow{a}\in A\in\mathbb{V}$, $p(\overrightarrow{a})\leq\alpha(\overrightarrow{a})$, i.e. $\alpha(\overrightarrow{a})$ is a term definable top element in $\mathbf{A}$. Conversely, if every $\mathbf{A}\in\mathbb{V}$ has a uniformly term definable top element $\alpha^{\mathbf{A}}(\overrightarrow{a})$, then clearly $\vDash_3^{\mathbb{V}}\alpha$. ■

Corollary 10 *(i) $\vDash_3^{\mathbb{A}}$ has no theorems; (ii) $\vDash_1^{\mathbb{A}}$ and $\vDash_2^{\mathbb{A}}$have the same theorems in the $\langle +,-,0\rangle$-fragment.*

Proof. (i) No nontrivial Abelian ℓ-group has a top element. (ii) If $\vDash_1^{\mathbb{A}}\alpha$, then clearly $\vDash_2^{\mathbb{A}}\alpha$. If $\vDash_2^{\mathbb{A}}\alpha$ and α does not contain lattice connectives, then given

$\overrightarrow{a} \in Z$, either $\alpha(\overrightarrow{a}) = 0$ (and we are home), or $\alpha(\overrightarrow{a}) > 0$, in which case there are $\overrightarrow{b} \in Z$ s.t. $\alpha\left(\overrightarrow{b}\right) < 0$, against the hypothesis[2]. ■

We now restrict our attention to e-ℓ-varieties which have suitable candidates for *implication* or *equivalence* connectives. Since the languages we are working with need not contain anything like a fusion connective with respect to which we may require that a candidate implication should behave as a residual, by means of the next definition we try to get as close as we can to the residuation requirement given the expressive means we have at our disposal.

Definition 11 *Let* $\mathbb{V}$ *be a e-ℓ-variety of type* $\mathcal{L}$*. We say that* $\mathbb{V}$ *has an* implication *iff there is* $\alpha(p,q) \in \mathtt{Fm}(\mathcal{L})$ *s.t.*

$$\mathbb{V} \vDash e \leq \alpha(p,q) \Leftrightarrow p \leq q.$$

We say that $\mathbb{V}$ *has an* equivalence *iff there is* $\alpha(p,q) \in \mathtt{Fm}(\mathcal{L})$ *s.t.*

$$\mathbb{V} \vDash e \approx \alpha(p,q) \Leftrightarrow p \approx q.$$

Finally, if $\mathbb{V}$ *has both an implication* $\beta(p,q)$ *and an equivalence* $\alpha(p,q)$*, we say that* α *is* well-behaved *w.r.t* β *iff*

$$\mathbb{V} \vDash \alpha(p,q) \leq \beta(p,q).$$

Implications may have additional properties of interest. A few are listed in the next table.

limit	$\mathbb{V} \vDash e \approx \alpha(p,q) \Leftrightarrow p \leq q$
e-standard	$\mathbb{V} \vDash p \approx \alpha(e,p)$
semisubtractive	$\mathbb{V} \vDash e \approx \alpha(p,p)$
subtractive	semisubtractive + e-standard
strongly subtractive	limit + e-standard
weakly suffixing	$\mathbb{V} \vDash p \leq q \Rightarrow \alpha(q,r) \leq \alpha(p,r)$
weakly prefixing	$\mathbb{V} \vDash p \leq q \Rightarrow \alpha(r,p) \leq \alpha(r,q)$
cancellative	$\mathbb{V} \vDash \alpha(p,q) \approx \alpha(p,r) \Rightarrow q \approx r$

We observe the following facts:

- If $\mathbb{V}$ has an equivalence $\alpha(p,q)$, the singleton $\{\alpha(p,q)\}$ is necessarily a system of protoequivalence formulas for $\vDash_1^{\mathbb{V}}$, while it yields such a system for $\vDash_2^{\mathbb{V}}$ provided $\mathbb{V}$ also has an implication $\beta(p,q)$ with respect to which α is well-behaved.
- Whenever $\mathbb{V}$ has a subtractive implication $\alpha(p,q)$, the formula $\alpha(q,p)$ witnesses subtractivity for $\mathbb{V}$ (whence the name we have chosen), even though the lack of a subtractive implication does not entail the failure of subtractivity.

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[2] This fact corresponds to the observation, made both in [26] and in [6], that the semantics of the intensional fragment of Abelian logic (viewed as a collection of theorems, not as a consequence relation) can also be given in terms of Abelian *groups*, rather than in terms of Abelian ℓ-groups.

- Every limit implication is semisubtractive by reflexivity of the lattice order, whence any strongly subtractive implication is subtractive.
- Any semisubtractive and cancellative implication is an equivalence.

We now give some examples of the preceding concepts.

Example 12 *The variety* $\mathbb{BA}$ *of Boolean algebras has a strongly subtractive implication* $p' \vee q$*, as well as an equivalence* $(p' \vee q) \wedge (q' \vee p)$ *which is well-behaved w.r.t. it.*

Example 13 *The variety* $\mathbb{MV}$ *of MV algebras has a strongly subtractive implication* $p' \oplus q$*, as well as an equivalence* $(p' \oplus q) \wedge (q' \oplus p)$ *which is well-behaved w.r.t. it.*

Example 14 *The variety* $\mathbb{OML}$ *of orthomodular lattices has a strongly subtractive implication* $p' \vee (p \wedge q)$ *(the* Sasaki hook*: [9]).*

As illustrated by the above examples, whenever e realises the top element of the lattice reduct in all algebras of $\mathbb{V}$, any implication in $\mathbb{V}$ is necessarily a limit implication. However, we may have limit implications even if the "$e = \top$" condition is not fulfilled. In particular, whenever $\mathbb{V}$ has an implication $p \rightarrow q$, it is easily seen that it also has a limit implication $e \wedge (p \rightarrow q)$. For example:

Example 15 *The variety* $\mathbb{A}$ *of Abelian* ℓ*-groups has a subtractive and cancellative implication* $q - p$ *which is, therefore, also an equivalence (and, of course, is well-behaved w.r.t. itself). The implication* $(q - p) \wedge 0$ *is not e-standard but is limit.*

Example 16 *The variety* $\mathbb{RL}$ *of residuated lattices has two primitive e-standard implications,* $p \backslash q$ *and* q/p*, neither of which is limit; the formulas* $e \wedge (p \backslash q)$ *and* $e \wedge (q/p)$ *are limit implications. The formulas* $(p \backslash q) \wedge (q \backslash p)\,, (p/q) \wedge (q/p)$ *are well-behaved equivalences (w.r.t. the corresponding implications) in a subvariety* $\mathbb{V}$ *of residuated lattices iff* $\mathbb{V} \vDash p/p \approx e \approx p \backslash p$.

Other interesting examples of e-ℓ-varieties with implication include:

Example 17 *The class of* $\{\wedge, \vee, \rightarrow, e\}$*-subreducts of commutative residuated lattices is an e-*ℓ*-variety with implication by Proposition 7.1 in [2]. Even though an explicit axiomatisation for this variety is not given in [2], it must satisfy at least the following equational conditions which need not hold in an arbitrary e-*ℓ*-variety with implication:*

$$\begin{array}{rl}
(i) & x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z); \\
(ii) & x \rightarrow (y \rightarrow z) \leq y \rightarrow (x \rightarrow z) \\
(iii) & e \rightarrow x \approx x \\
(iv) & (x \rightarrow y) \wedge (x \rightarrow z) \approx x \rightarrow y \wedge z \\
(v) & x \leq (x \rightarrow y) \wedge z \rightarrow y \\
(vi) & x \leq ((e \vee z) \rightarrow (x \rightarrow y)) \rightarrow y
\end{array}$$

Example 18 *The variety generated by the class of* $\{\wedge, \vee, \rightarrow, e\}$*-subreducts of positive Ackermann groupoids, used in [25] as models of the minimal relevant logic* $\mathbf{B}^{+}$*, is an e-*ℓ*-variety with implication. To the best of our knowledge, nothing is known as regards an equational basis for this variety.*

In the next Definition we list some other sensible connectives with implicative features which arise in the languages of e-ℓ-varieties with implication.

Definition 19 *We define:*

$$\begin{aligned} \alpha \supset \beta &= \alpha \wedge e \rightarrow \beta; \\ \alpha \rightsquigarrow \beta &= e \wedge (\alpha \rightarrow \beta)\,; \\ \alpha \supset_M \beta &= \alpha \wedge e \rightarrow \beta \vee e. \end{aligned}$$

The connective $\supset$ is termed *enthymematic*, or *intuitionistic*, implication in the tradition of relevance logics; in the context of Abelian logic, it was used in [26] to embed positive Łukasiewicz logic into Abelian logic. The other two implication connectives, $\rightsquigarrow$ and $\supset_M$, were alternatively employed in [23] and [24] to embed full Łukasiewicz logic into Abelian logic. In the more general setting, the table below shows to what extent modus ponens holds for each of these implications in the logics we consider (Y: modus ponens holds; N: modus ponens fails in general; S: modus ponens holds provided $\rightarrow$ is subtractive in $\mathbb{V}$).

	$\vDash_1^{\mathbb{V}}$	$\vDash_2^{\mathbb{V}}$	$\vDash_3^{\mathbb{V}}$
$p, p \rightarrow q \vDash q$	S	Y	N
$p, p \supset q \vDash q$	S	Y	N
$p, p \rightsquigarrow q \vDash q$	N	Y	N
$p, p \supset_M q \vDash q$	N	N	N

In view of the results to be proved in the next section, it is convenient to notice the following fact:

Lemma 20 *Let* $\mathbb{V}$ *be a e-*ℓ*-variety of type* $\mathcal{L}$ *with implication* $\rightarrow$ *and equivalence* $\leftrightarrow$*.*

(i) The following are equivalent for $\alpha, \beta, \gamma_i, \sigma_i \in \mathtt{Fm}(\mathcal{L})$*:*

1. $\{\gamma_i \leftrightarrow \sigma_i : i \leq n\} \vDash_1^{\mathbb{V}} \alpha \leftrightarrow \beta$;
2. $\{\gamma_i \rightarrow \sigma_i, \sigma_i \rightarrow \gamma_i : i \leq n\} \vDash_2^{\mathbb{V}} \{\alpha \rightarrow \beta, \beta \rightarrow \alpha\}$;
3. $\{\gamma_i \approx \sigma_i : i \leq n\} \vDash_{\mathtt{Eq}(\mathbb{V})} \alpha \approx \beta$.

   (ii) Moreover, if $\rightarrow$ *is either limit, or both semisubtractive and cancellative, all of the previous items are equivalent to:*

4. $\{\gamma_i \rightarrow \sigma_i, \sigma_i \rightarrow \gamma_i : i \leq n\} \vDash_1^{\mathbb{V}} \{\alpha \rightarrow \beta, \beta \rightarrow \alpha\}$.

Proof. (i) 1 ⇔ 3. By definition, $\{\gamma_i \leftrightarrow \sigma_i : i \leq n\} \vDash_1^{\mathbb{V}} \alpha \leftrightarrow \beta$ means that

$$\{\gamma_i \leftrightarrow \sigma_i \approx e : i \leq n\} \vDash_{\mathtt{Eq}(\mathbb{V})} \alpha \leftrightarrow \beta \approx e.$$

Since $\leftrightarrow$ is an equivalence, however, we have our conclusion.

3 ⇒ 2. If 3. holds and $\overrightarrow{a} \in A \in \mathbb{V}$ are s.t. for every $i \leq n$, $e \leq \gamma_i \to \sigma_i(\overrightarrow{a}), \sigma_i \to \gamma_i(\overrightarrow{a})$, we have that for every $i \leq n$, $\gamma_i(\overrightarrow{a}) \leq \sigma_i(\overrightarrow{a})$ and $\sigma_i(\overrightarrow{a}) \leq \gamma_i(\overrightarrow{a})$,whence for every $i \leq n$, $\gamma_i(\overrightarrow{a}) = \sigma_i(\overrightarrow{a})$. Our hypothesis then implies $\alpha(\overrightarrow{a}) = \beta(\overrightarrow{a})$, which entails that $\alpha(\overrightarrow{a}) \leq \beta(\overrightarrow{a})$ and $\beta(\overrightarrow{a}) \leq \alpha(\overrightarrow{a})$, whence $e \leq \alpha \to \beta(\overrightarrow{a}), \beta \to \alpha(\overrightarrow{a})$.

2 ⇒ 1. Suppose that 2. holds and that $\overrightarrow{a} \in A \in \mathbb{V}$ are s.t. for every $i \leq n$, $e = \gamma_i \leftrightarrow \sigma_i(\overrightarrow{a})$. Then for every i, $\gamma_i(\overrightarrow{a}) = \sigma_i(\overrightarrow{a})$, whence $e \leq \gamma_i \to \sigma_i(\overrightarrow{a}), \sigma_i \to \gamma_i(\overrightarrow{a})$. It follows that $\alpha(\overrightarrow{a}) \leq \beta(\overrightarrow{a})$ and $\beta(\overrightarrow{a}) \leq \alpha(\overrightarrow{a})$, whence $\alpha(\overrightarrow{a}) = \beta(\overrightarrow{a})$, i.e. $e = \alpha(\overrightarrow{a}) \leftrightarrow \beta(\overrightarrow{a})$.

(ii) 3 ⇔ 4. Suppose that 3. holds and that$\overrightarrow{a} \in A \in \mathbb{V}$ are s.t. for every $i \leq n$, $\gamma_i \to \sigma_i(\overrightarrow{a}) = e = \sigma_i \to \gamma_i(\overrightarrow{a})$. If the implication is limit, it follows that for every $i \leq n$, $\gamma_i(\overrightarrow{a}) \leq \sigma_i(\overrightarrow{a})$ and $\sigma_i(\overrightarrow{a}) \leq \gamma_i(\overrightarrow{a})$, whence as in the first part of the proof $\alpha(\overrightarrow{a}) \leq \beta(\overrightarrow{a})$ and $\beta(\overrightarrow{a}) \leq \alpha(\overrightarrow{a})$, whereby $\alpha \to \beta(\overrightarrow{a}) = e = \beta \to \alpha(\overrightarrow{a})$ using again the limit assumption. Conversely, if $\mathbb{V} \vDash \{\gamma_i \approx \sigma_i : i \leq n\}$, once we take into account the fact that we have a limit arrow, the proof goes through exactly like for the implication 2 ⇒ 1.

If the implication is both semisubtractive and cancellative, then for every $i \leq n$, $\gamma_i \to \gamma_i(\overrightarrow{a}) = e = \gamma_i \to \sigma_i(\overrightarrow{a})$ and so $\gamma_i(\overrightarrow{a}) = \sigma_i(\overrightarrow{a})$. Then again $\alpha(\overrightarrow{a}) = \beta(\overrightarrow{a})$ and so

$$\beta \to \alpha(\overrightarrow{a}) = \beta \to \beta(\overrightarrow{a}) = e = \alpha \to \alpha(\overrightarrow{a}) = \alpha \to \beta(\overrightarrow{a}).$$

■

3 Algebraisation and deductive filters

A significant problem we should tackle when considering our logics from the viewpoint of AAL is where they sit within the so-called Leibniz hierarchy. In case $\mathbb{V}$ is an e-regular variety (for example, in case $\mathbb{V} = \mathbb{A}$) the issue is easily disposed of as regards $\vDash_1^{\mathbb{V}}$: due to well-known results, the assertional logic of $\mathbb{V}$ must be regularly algebraisable with $\mathbb{V}$ as an equivalent algebraic semantics. In the general case, the next theoerm provides an answer whenever $\mathbb{V}$ has both an implication and an equivalence.

Theorem 21 *Let $\mathbb{V}$ be a e-ℓ-variety of type $\mathcal{L}$ with implication $\to$ and equivalence $\leftrightarrow$.*

1. *$\vDash_1^{\mathbb{V}}$ is strongly and finitely algebraisable with $\mathbb{V}$ as an equivalent algebraic semantics. A system $\Delta(p,q)$ of equivalence formulas is given by the singleton $\{p \leftrightarrow q\}$; a system $E(p)$ of defining equations is given by the singleton $\{p \approx e\}$.*

2. *If $\to$ is either limit, or both semisubtractive and cancellative, $\{p \to q, q \to p\}$ is an alternative system of equivalence formulas.*

3. *If $\mathbb{V}$ is e-regular, $\vDash_1^{\mathbb{V}}$is also regularly algebraisable.*

Proof.

1. It suffices to show that (i) p is $\vDash_1^{\mathbb{V}}$-interderivable with $\Delta\left(E(p)\right)$; (ii) the equational consequence relation of $\mathbb{V}$ is interpretable into $\vDash_1^{\mathbb{V}}$by means of $\{p \leftrightarrow q\}$. (i) p is $\vDash_1^{\mathbb{V}}$-interderivable with $p \leftrightarrow e$: if $a \in A \in \mathbb{V}$ is such that $p^{\mathbf{A}}\left(a\right) = e^{\mathbf{A}}$, then by definition $e^{\mathbf{A}} = p^{\mathbf{A}}\left(a\right) \leftrightarrow e^{\mathbf{A}}$; the converse is also a matter of applying definitions. (ii) Suppose $\{\gamma_i \approx \sigma_i : i \leq n\} \vDash_{\mathtt{Eq}(\mathbb{V})} \alpha \approx \beta$. By Lemma 20.(i),

$$\{\gamma_i \leftrightarrow \sigma_i : i \leq n\} \vDash_1 \alpha \leftrightarrow \beta, \text{ i.e. } \{\Delta(\gamma_i, \sigma_i) : i \leq n\} \vDash_1 \Delta\left(\alpha, \beta\right).$$

   The converse also follows from the same Lemma.

2. (i) For $\mathbf{A} \in \mathbb{V}$, $e^{\mathbf{A}} \leq e^{\mathbf{A}}$ implies $e^{\mathbf{A}} = e^{\mathbf{A}} \to e^{\mathbf{A}}$ by the subtractivity of the arrow, whereas, if $a \in A$ and $p^{\mathbf{A}}\left(a\right) \to e^{\mathbf{A}} = e^{\mathbf{A}} = e^{\mathbf{A}} \to p^{\mathbf{A}}\left(a\right)$, then $e^{\mathbf{A}} \leq p^{\mathbf{A}}\left(a\right), p^{\mathbf{A}}\left(a\right) \leq e^{\mathbf{A}}$ and thus $p^{\mathbf{A}}\left(a\right) = e^{\mathbf{A}}$. Therefore p is $\vDash_1^{\mathbb{V}}$-interderivable with $\{p \to e, e \to p\}$. (ii) By Lemma 20.(ii), the equational consequence relation of $\mathbb{V}$ is interpretable into $\vDash_1^{\mathbb{V}}$by means of $\{p \to q, q \to p\}$ if the arrow abides by one of the indicated conditions.

3. As already remarked, this follows from the fact that $\vDash_1^{\mathbb{V}}$is the e-assertional logic of $\mathbb{V}$.

■

Corollary 22 *([18])$\vDash_1^{\mathbb{A}}$ is regularly, strongly and finitely algebraisable with $\mathbb{A}$ as an equivalent algebraic semantics. A system $\Delta(p,q)$ of equivalence formulas is given either by the singleton $\{q - p\}$ or by the pair $\{q - p, p - q\}$; a system $E(p)$ of defining equations is given by the singleton $\{p \approx 0\}$.*

The next result illustrates the interesting fact that $\vDash_1^{\mathbb{V}}$ and $\vDash_2^{\mathbb{V}}$ are different algebraisable logics with the same equivalent algebraic semantics, a well-known but not too usual phenomenon in AAL. In the terminology of [5], the next theorem can be rephrased in the following terms: if $\mathbb{V}$ is a e-ℓ-variety of type $\mathcal{L}$ with implication $\to$, then $\mathbb{V}$ is $\langle e, e \wedge a\rangle$-regular and $\vDash_2^{\mathbb{V}} = \mathcal{S}(\mathbb{V}, \langle e, e \wedge a\rangle\ )$.

Theorem 23 *Let $\mathbb{V}$ be a e-ℓ-variety of type $\mathcal{L}$ with implication $\to$. $\vDash_2^{\mathbb{V}}$ is finitely and strongly algebraisable with $\mathbb{V}$ as an equivalent algebraic semantics. A system $\Delta(p,q)$ of equivalence formulas is given by the set $\{p \to q, q \to p\}$; a system $E(p)$ of defining equations is given by the singleton $\{p \wedge e \approx e\}$.*

Proof. To prove $\vDash_2^{\mathbb{V}}$ algebraisable, once again it suffices to show that (i) p is $\vDash_2$-interderivable with $\Delta\left(E(p)\right)$; (ii) the equational consequence relation of $\mathbb{V}$ is interpretable into $\vDash_2^{\mathbb{V}}$ by means of $\{p \to q, q \to p\}$.

(i) What we must prove is that p is $\vDash_2^{\mathbb{V}}$-interderivable with $\{p \supset e, e \rightarrow (p \wedge e)\}$. In fact, if $e \leq a \in A \in \mathbb{V}$, then $e \leq a \wedge e$ and so $e \leq e \rightarrow a \wedge e$, while $e \leq (a \wedge e) \rightarrow e$ holds unconditionally for $a \wedge e \leq e$. Conversely, if $e \leq e \rightarrow a \wedge e$, then $e \leq a \wedge e \leq a$.

(ii) Suppose $\{\gamma_i \approx \sigma_i : i \leq n\} \vdash_{\mathtt{Eq}(\mathbb{V})} \alpha \approx \beta$. By Lemma 20.(i),

$$\{\gamma_i \rightarrow \sigma_i, \sigma_i \rightarrow \gamma_i : i \leq n\} \vDash_2^{\mathbb{V}} \{\alpha \rightarrow \beta, \beta \rightarrow \alpha\}, \text{ i.e. } \{\Delta(\gamma_i, \sigma_i) : i \leq n\} \vDash_2^{\mathbb{V}} \Delta(\alpha, \beta).$$

The converse also follows from the same Lemma. ■

Corollary 24 *(i) $\vDash_2^{\mathbb{A}}$ is strongly and finitely algebraisable with $\mathbb{A}$ as an equivalent algebraic semantics. A system $\Delta(p, q)$ of equivalence formulas is given by the pair $\{q - p, p - q\}$; a system $E(p)$ of defining equations is given by the singleton $\{p \wedge 0 \approx 0\}$. (ii) $\vDash_2^{\mathbb{A}}$ is not regularly algebraisable.*

Proof. (i) follows from Theorem 23. (ii) To see that $\vDash_2^{\mathbb{A}}$ is not regularly algebraisable, it is enough to give a counterexample to the Gödel rule $p, q \vDash_2^{\mathbb{A}} \Delta(p, q)$. This counterexample is easily found in the integers, e.g. by assigning the variable p the value 2 and the variable q the value 1. ■

The logic $\vDash_3^{\mathbb{V}}$ ranks much lower in the Leibniz hierarchy, as certified by the following proposition:

Theorem 25 *There exist e-ℓ-varieties $\mathbb{V}$ s.t. $\vDash_3^{\mathbb{V}}$ is not protoalgebraic. In particular, $\vDash_3^{\mathbb{A}}$ is an example.*

Proof. Font et al. [11] had already proved that $\vDash_3^{\mathbb{MV}}$ is not protoalgebraic. As regards our claim concerning $\vDash_3^{\mathbb{A}}$, this logic fails to be almost inconsistent, since e.g. $\alpha + \alpha \nvDash_3^{\mathbb{A}} \alpha$. By Corollary 10 and Lemma 4.12 in [21], then, it is not protoalgebraic. ■

Of course, there are varieties for which $\vDash_3^{\mathbb{V}}$ is distinct from $\vDash_i^{\mathbb{V}}$ $(i = 1, 2)$, yet finitely equivalential: take for example $\mathbb{V}$ to be the variety of interior algebras ([7], Chapter 3). In the light of the preceding general result, the logics $\vDash_3^{\mathbb{V}}$ fail to be as tightly linked to the varieties $\mathbb{V}$ they arise out of, in that such varieties need not be equivalent algebraic semantics for them. In an important sense, though, there is a strong connection: the algebra reducts of matrix models of $\vDash_3^{\mathbb{V}}$ are exactly the members of $\mathbb{V}$.

Lemma 26 *Let $\mathbb{V}$ be a e-ℓ-variety. (i) $\vDash_3^{\mathbb{V}}$ is selfextensional and has the conjunction property; (ii) $\mathbf{Alg}\left(\vDash_3^{\mathbb{V}}\right) = \mathbb{V}$.*

Proof. (i) α is $\vDash_3^{\mathbb{V}}$-interderivable with β just in case $\mathbb{V} \vDash \alpha \approx \beta$, whence $\vDash_3^{\mathbb{V}}$ is certainly selfextensional. The formula $\alpha \wedge \beta$ witnesses the conjunction property. (ii) In virtue of (i) and of Lemma 2.43 in [12], $\mathbf{Alg}\left(\vDash_3^{\mathbb{V}}\right)$ must be a variety which is defined by

$$\left\{\alpha \approx \beta : \alpha, \beta \text{ are } \vDash_3^{\mathbb{V}} \text{-interderivable}\right\} = \{\alpha \approx \beta : \mathbb{V} \vDash \alpha \approx \beta\}$$

i.e. $\mathbf{Alg}\left(\vDash_3^{\mathbb{V}}\right) = \mathbb{V}$. ■

What can we say about the deductive filters of logics from e-ℓ-varieties? As regards $\vDash_1^{\mathbb{V}}$, we have an expected characterisation which applies in case $\mathbb{V}$ is e-regular and has a subtractive implication (expected to the extent that these two conditions jointly imply that $\mathbb{V}$ is ideal determined).

Lemma 27 *Let $\mathbb{V}$ be an e-regular e-ℓ-variety with subtractive implication $\rightarrow$, and let $\mathbf{A} \in \mathbb{V}$. The $\vDash_1^{\mathbb{V}}$-filters on $\mathbf{A}$ are its Ursini ideals.*

Proof. Under these assumptions $\mathbb{V}$ is e-subtractive, witness the term $y \rightarrow x$. By results in [1], e-subtractivity implies that the deductive filters of its e-assertional logic on the algebra reducts of the logic's matrix models (which amount to members of $\mathbb{V}$, by e-regularity) are its Ursini ideals. ■

Corollary 28 *([18]) Let $\mathbf{A}$ be an Abelian ℓ-group. The $\vDash_1^{\mathbb{A}}$-filters on $\mathbf{A}$ are its convex ℓ-subgroups.*

Proof. We simply take advantage of the fact that $\mathbb{A}$ is an ideal determined variety and that there is a 1-1 correspondence between congruences and convex ℓ-subgroups. ■

$\vDash_3^{\mathbb{V}}$-filters on members of $\mathbb{V}$ can be identified even more easily.

Lemma 29 *Let $\mathbb{V}$ be an e-ℓ-variety, and let $\mathbf{A} \in \mathbb{V}$. The $\vDash_3^{\mathbb{V}}$-filters on $\mathbf{A}$ are the filters of its lattice reduct.*

Proof. Let $\mathbf{A} \in \mathbb{V}$, and let F be any of its lattice filters. Suppose moreover that $\Gamma \vDash_3^{\mathbb{V}} \alpha$ and that $\overrightarrow{a} \in A$ are such that $\left\{\gamma^{\mathbf{A}}\left(\overrightarrow{a}\right) : \gamma \in \Gamma\right\} \subseteq F$. Then $\bigwedge^{\mathbf{A}} \left\{\gamma^{\mathbf{A}}\left(\overrightarrow{a}\right) : \gamma \in \Gamma\right\} \in F$, whence $\bigwedge^{\mathbf{A}} \left\{\gamma^{\mathbf{A}}\left(\overrightarrow{a}\right) : \gamma \in \Gamma\right\} \leq^{\mathbf{A}} \alpha^{\mathbf{A}}\left(\overrightarrow{a}\right)$. Consequently, $\alpha^{\mathbf{A}}\left(\overrightarrow{a}\right) \in F$.

Conversely, closure w.r.t. meets follows from the fact that $\alpha, \beta \vDash_3^{\mathbb{V}} \alpha \wedge \beta$. On the other hand, it is readily seen that $\alpha \vDash_3^{\mathbb{V}} \beta$ iff $\mathbb{V} \vDash \alpha \leq \beta$. This shows that $\vDash_3^{\mathbb{V}}$-filters on members of $\mathbb{V}$ must be upward closed. ■

As regards $\vDash_2^{\mathbb{V}}$, we focus on the special case of e-ℓ-varieties with an implication and with compatible operations. This concept is clarified in the next

Definition 30 *Let $\mathbb{V}$ be an e-ℓ-variety with an implication $\rightarrow$. We say that $\mathbb{V}$ has* compatible operations *iff, for any $\mathbf{A} \in \mathbb{V}$ and for any functor f in the type not belonging to $\{\wedge, \vee, \rightarrow, e\}$, every congruence on the reduct $\langle A, \wedge, \vee, \rightarrow, e\rangle$ has the substitution property w.r.t. f.*

Theorem 31 *Let $\mathbb{V}$ be an e-ℓ-variety with an e-standard, weakly suffixing and weakly prefixing implication $\rightarrow$, and with compatible operations. Let $\mathbf{A}$ be an algebra of the same type as $\mathbb{V}$. Then $F \subseteq A$ is a $\vDash_2^{\mathbb{V}}$-filter on $\mathbf{A}$ iff the following conditions are satisfied for all $a, b, c, \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$:*

F1 *$e \in F$;*

F2 *$a \in F, a \leq b \Rightarrow b \in F$;*

F3 *$a, a \to b \in F \Rightarrow b \in F$;*

F4' *$a \to b, a \to c \in F \Rightarrow a \to b \wedge c \in F$;*

F4" *$a \to c, b \to c \in F \Rightarrow a \vee b \to c \in F$;*

F5 *$a \to b, c \to d \in F \Rightarrow (b \to c) \to (a \to d) \in F$;*

F6 *$\{a_i \to b_i, b_i \to a_i\}_{i \leq n} \subseteq F \Rightarrow \left\{ f(\overrightarrow{a}, \overrightarrow{c}) \to f\left(\overrightarrow{b}, \overrightarrow{c}\right), f\left(\overrightarrow{b}, \overrightarrow{c}\right) \to f(\overrightarrow{a}, \overrightarrow{c}) \right\}$ $\subseteq F$, for every functor f in the type not belonging to $\{\wedge, \vee, \to, e\}$.*

We need to prove a few claims in order to prove the theorem.

Claim 32 *Given F3-F6, conditions F1 and F2 are jointly equivalent to*

$$\text{(F7)}\ \vDash_2^{\mathbb{V}} \alpha \Rightarrow \alpha^{\mathbf{A}}(\overrightarrow{a}) \in F \text{ for any } \overrightarrow{a} \in A.$$

Proof. If F1 and F2 hold, the positive cone $P^{\mathbf{A}}$ is contained in F, whence we get our conclusion. Conversely, let F7 hold. Then F1 is satisfied since $\vDash_2^{\mathbb{V}} e$; moreover, let $\alpha^{\mathbf{A}}(\overrightarrow{a}) \in F$ and $\alpha^{\mathbf{A}}(\overrightarrow{a}) \leq \beta^{\mathbf{A}}(\overrightarrow{a})$. Then $e \leq \alpha^{\mathbf{A}}(\overrightarrow{a}) \to \beta^{\mathbf{A}}(\overrightarrow{a})$, whereby $\vDash_2^{\mathbb{V}} \alpha \to \beta$, as $\overrightarrow{a}$ was an arbitrary n-tuple. By F7, $\alpha^{\mathbf{A}}(\overrightarrow{a}) \to \beta^{\mathbf{A}}(\overrightarrow{a}) \in F$, and by F3, $\beta^{\mathbf{A}}(\overrightarrow{a}) \in F$. ■

Claim 33 *Let $\mathbf{A}$ be an algebra of the same type as $\mathbb{V}$, and let F be a $\vDash_2^{\mathbb{V}}$-filter on $\mathbf{A}$. Then F is closed w.r.t. the following rules:*

$$\begin{array}{ll} \text{(F8)} & a, b \in F \Rightarrow a \wedge b \in F; \\ \text{(F9)} & a \to b, b \to c \in F \Rightarrow a \to c \in F. \end{array}$$

Proof. Suppose that $a, b \in F$. Since the implication is e-standard, $e \to a, e \to b \in F$. By F4', $e \to a \wedge b \in F$. So, by F1 and F3, $a \wedge b \in F$, and this takes care of F8. As regards F9, suppose that $a \to b, b \to c \in F$. Then, F5 entails that $(b \to b) \to (a \to c) \in F$; but F7 dictates that $b \to b \in F$, whence by F3 $a \to c \in F$. ■

Now we have all the ingredients we need to prove our main theorem.

Proof. By Claim 32, it is enough to prove that every deductive system $\langle \mathcal{L}(\mathbb{V}), \vdash \rangle$ which contains the rules

R1 $\vdash \alpha$, for $\vDash_2^{\mathbb{V}} \alpha$;

R2 $\alpha, \alpha \to \beta \vdash \beta$;

R3' $\alpha \to \beta, \alpha \to \gamma \vdash \alpha \to \beta \wedge \gamma$;

R3" $\alpha \to \gamma, \beta \to \gamma \vdash \alpha \vee \beta \to \gamma$;

R4 $\alpha \to \beta, \gamma \to \delta \vdash (\beta \to \gamma) \to (\alpha \to \delta)$;

R5 $\{\alpha_i \to \beta_i, \beta_i \to \alpha_i\}_{i \leq n} \vdash \left\{ f(\overrightarrow{\alpha}, \overrightarrow{\gamma}) \to f\left(\overrightarrow{\beta}, \overrightarrow{\gamma}\right), f\left(\overrightarrow{\beta}, \overrightarrow{\gamma}\right) \to f(\overrightarrow{\alpha}, \overrightarrow{\gamma}) \right\}$, for every functor f in the type not belonging to $\{\wedge, \vee, \to, e\}$

is algebraisable with $\mathbb{V}$ as equivalent algebraic semantics, with $\Delta = \{p \to q, q \to p\}$ as a system of equivalence formulas, and with $\{e \approx e \wedge p\}$ as a set of defining equations. Once this is done, the conclusion follows from Theorem 23.

To show algebraisability with the indicated system of equivalence formulas and set of defining equations, it suffices to prove: i) $\vdash \alpha \to \alpha$; ii) $\alpha \to \beta, \beta \to \gamma \vdash \alpha \to \gamma$; iii) $\alpha \to \beta, \gamma \to \delta \vdash \{\alpha \wedge \gamma \to \beta \wedge \delta, \alpha \vee \gamma \to \beta \vee \delta\}$; iv) $\alpha \to \beta, \gamma \to \delta \vdash (\beta \to \gamma) \to (\alpha \to \delta)$; v) $\{\alpha_i \to \beta_i, \beta_i \to \alpha_i\}_{i \leq n} \vdash \left\{f\left(\overrightarrow{\alpha}, \overrightarrow{\gamma}\right) \to f\left(\overrightarrow{\beta}, \overrightarrow{\gamma}\right), f\left(\overrightarrow{\beta}, \overrightarrow{\gamma}\right) \to f\left(\overrightarrow{\alpha}, \overrightarrow{\gamma}\right)\right\}$, for every functor f in the type not belonging to $\{\wedge, \vee, \to, e\}$; (vi) $\alpha \dashv\vdash \{e \to \alpha \wedge e, \alpha \wedge e \to e\}$.

We readily observe that iv) and v) correspond to R4 and R5; i) follows from R1, while ii) is proved like in Claim 33. So, we are left with the task of proving iii) and vi).

As regards iii), we only prove that $\alpha \to \beta, \gamma \to \delta \vdash \alpha \wedge \gamma \to \beta \wedge \delta$. Now, $\mathbb{V} \vDash \alpha \wedge \gamma \leq \alpha$ implies that $\mathbb{V} \vDash \alpha \to \beta \leq \alpha \wedge \gamma \to \beta$, since our implication is weakly suffixing. This means that $\vDash_2^{\mathbb{V}} (\alpha \to \beta) \to (\alpha \wedge \gamma \to \beta)$, whence by R1 $\vdash (\alpha \to \beta) \to (\alpha \wedge \gamma \to \beta)$. Then, by R2, $\alpha \to \beta \vdash \alpha \wedge \gamma \to \beta$. Applying R1 once again, we also get $\alpha \to \beta \vdash \alpha \wedge \gamma \to \gamma$. Now we resort to R3 and we obtain $\alpha \to \beta \vdash \alpha \wedge \gamma \to \beta \wedge \gamma$. Similarly, $\gamma \to \delta \vdash \beta \wedge \gamma \to \beta \wedge \delta$. By ii), then, we attain the desired conclusion.

As regards vi), we have that:

1. $\vdash \alpha \wedge e \to e$ (R1)
2. $\alpha \vdash e \to \alpha$ (R1, R2, e-standard hyp.)
3. $e \to \alpha \vdash e \to \alpha \wedge e$ (R1, R3)
4. $\alpha \vdash e \to \alpha \wedge e$ (2, 3)

Conversely,

1. $\vdash e$ (R1)
2. $e \to \alpha \wedge e \vdash \alpha \wedge e$ (1, R2)
3. $\vdash \alpha \wedge e \to \alpha$ (R1)
4. $e \to \alpha \wedge e \vdash \alpha$ (2, 3, R2)

What we still need to prove is that $\mathbb{V}$ is the equivalent algebraic semantics of $\langle \mathcal{L}(\mathbb{V}), \vdash \rangle$. By Theorem 2.17 in [3], such an algebraic semantics can be axiomatised by means of the following quasiequations:

Q1 $e \approx e \wedge (p \to p)$
Q2 $e \approx e \wedge \alpha$, if $\vDash_2^{\mathbb{V}} \alpha$
Q3 $e \approx e \wedge p \,\&\, e \approx e \wedge (p \to q) \Rightarrow e \approx e \wedge q$
Q4' $e \approx e \wedge (p \to q) \,\&\, e \approx e \wedge (p \to r) \Rightarrow e \approx e \wedge (p \to q \wedge r)$
Q4" $e \approx e \wedge (p \to r) \,\&\, e \approx e \wedge (q \to r) \Rightarrow e \approx e \wedge (p \vee q \to r)$
Q5 $e \approx e \wedge (p \to q) \,\&\, e \approx e \wedge (r \to s) \Rightarrow e \approx e \wedge ((q \to r) \to (p \to s))$
Q6 $\{e \approx e \wedge (p_i \to q_i), e \approx e \wedge (q_i \to p_i)\}_{i \leq n} \Rightarrow$ $\{e \approx e \wedge (f(\overrightarrow{p}, \overrightarrow{r}) \to f(\overrightarrow{q}, \overrightarrow{r})), e \approx e \wedge (f(\overrightarrow{q}, \overrightarrow{r}) \to f(\overrightarrow{p}, \overrightarrow{r}))\}$, for every functor f in the type not belonging to $\{\wedge, \vee, \to, e\}$
Q7 $e \approx e \wedge (p \to q) \,\&\, e \approx e \wedge (q \to p) \Rightarrow p \approx q$

To obtain our goal, we must check that $\mathbb{V}$ satisfies all of Q1-Q7 and, conversely, that all the axioms of $\mathbb{V}$ follow from Q1-Q7. However, remark that

$$\begin{array}{ll} & \mathbb{V} \vDash e \approx e \wedge (\alpha_1 \rightarrow \beta_1) \ \& \ldots \& e \approx e \wedge (\alpha_n \rightarrow \beta_n) \Rightarrow e \approx e \wedge (\alpha \rightarrow \beta) \\ \text{iff} & \mathbb{V} \vDash \alpha_1 \leq \beta_1 \ \& \ldots \& \alpha_n \leq \beta_n \Rightarrow \alpha \leq \beta \end{array}$$

by the definitions of lattice order and implication. Recall that we assumed our arrow to be e-standard, whence $\mathbb{V}$ satisfies $e \wedge \alpha \approx e \wedge (e \rightarrow \alpha)$. Taking this fact into account, it follows that:

$\mathbb{V} \vDash$ Q1	iff $\mathbb{V} \vDash p \leq p$ (OK by reflexivity)
$\mathbb{V} \vDash$ Q2	iff $\mathbb{V} \vDash e \leq \alpha$ (OK if $\vDash_2^{\mathbb{V}} \alpha$)
$\mathbb{V} \vDash$ Q3	iff $\mathbb{V} \vDash e \leq p \ \& \ p \leq q \Rightarrow e \leq q$ (OK by transitivity)
$\mathbb{V} \vDash$ Q4'	iff $\mathbb{V} \vDash p \leq q \ \& p \leq r \Rightarrow p \leq q \wedge r$ (OK by lattice prop.)
$\mathbb{V} \vDash$ Q4"	analogous
$\mathbb{V} \vDash$ Q5	iff $\mathbb{V} \vDash p \leq q \ \& r \leq s \Rightarrow q \rightarrow r \leq p \rightarrow s$ (OK by the weakly suff. and weakly pref. hypothesis)
$\mathbb{V} \vDash$ Q6	iff $\mathbb{V} \vDash \{p_i \approx q_i\}_{i \leq n} \Rightarrow (f\,(p_1, ..., p_n, \overrightarrow{r}) \approx f\,(q_1, ..., q_n, \overrightarrow{r}))$ (OK by antisymmetry and compatible operations)
$\mathbb{V} \vDash$ Q7	iff $\mathbb{V} \vDash p \leq q \ \& q \leq p \Rightarrow p \approx q$ (OK by antisymmetry)

For the converse, just observe that, for any formulae α, β, if $\vdash \alpha \rightarrow \beta, \beta \rightarrow \alpha$, then the equivalent algebraic semantics satisfies both $e \approx e \wedge (\alpha \rightarrow \beta)$ and $e \approx e \wedge (\beta \rightarrow \alpha)$, whence by Q7 it satisfies $\alpha \approx \beta$ as well. So it suffices to prove that, if $\alpha \approx \beta$ is a valid identity of $\mathbb{V}$, then $\vdash \alpha \rightarrow \beta, \beta \rightarrow \alpha$. But this is easily seen: suppose in fact that $\mathbb{V} \vDash \alpha \approx \beta$. Then $\mathbb{V} \vDash \{\alpha \leq \beta, \beta \leq \alpha\}$, whence by the definition of implication $\mathbb{V} \vDash \{e \leq \alpha \rightarrow \beta, e \leq \beta \rightarrow \alpha\}$, i.e. $\vDash_2^{\mathbb{V}} \alpha \rightarrow \beta, \beta \rightarrow \alpha$. By R1, then, $\vdash \alpha \rightarrow \beta, \beta \rightarrow \alpha$. ■

As regards $\vDash_2^{\mathbb{V}}$, the previous theorem specialises to the following characterisation of $\vDash_2^{\mathbb{A}}$-filters on Abelian ℓ-groups (cp. the analogous characterisation of deductive filters of substructural logics on residuated lattices [17]).

Definition 34 *Let* **A** *be an Abelian ℓ-group. A subset F is called a* congruence filter *of* **A** *if it satisfies the following conditions:*

S1 $0 \in F$

S2 $a, b \in F \Rightarrow a + b \in F$

S3 $a \in F \Rightarrow a \wedge 0 \in F$

S4 $a \in F, a \leq b \Rightarrow b \in F$

Lemma 35 *Let* $\mathbf{A}$ *be an Abelian* ℓ*-group, and let* F *be a congruence filter of* $\mathbf{A}$*. If* $a, b \in F$*, then* $a \wedge b \in F$*.*

Proof. Suppose $a, b \in F$. Then, by S3, $a \wedge 0, b \wedge 0 \in F$ and, by S1, $(a \wedge 0) + (b \wedge 0) = (a + b) \wedge a \wedge b \wedge 0 \in F$. Finally, by S4 we get our conclusion. ■

Lemma 36 *Let* $\mathbf{A}$ *be an Abelian* ℓ*-group. The following are equivalent for* $F \subseteq A$*:*

1. *F is a* $\vDash_2^{\mathbb{A}}$*-filter;*
2. *F is a congruence filter;*
3. *there exists a convex* ℓ*-subgroup* J *of* $\mathbf{A}$ *s.t.* $F = \uparrow J$*.*

Proof. 1. implies 2. This follows from the fact that $\vDash_2^{\mathbb{A}} 0$, $\alpha, \beta \vDash_2^{\mathbb{A}} \alpha + \beta$ and $\alpha \vDash_2^{\mathbb{A}} \alpha \wedge 0$. Also, since $\vDash_3^{\mathbb{A}} \subseteq \vDash_2^{\mathbb{A}}$, F must be a lattice filter and thus upward closed.

2. implies 3. Let $J = \{a \in F : -a \in F\}$. Clearly, $F = \uparrow J$. All that remains to prove, therefore, is that J is a convex ℓ-subgroup of $\mathbf{A}$. If $a, b \in J$, then $a, b, -a, -b \in F$, whence $a + b$ and $-a + -b = -(a + b) \in F$, which means $a + b \in J$. If $a \in J$ and $|b| \leq |a|$, then on the one side $a \wedge -a \leq b \wedge -b \leq b, -b$, on the other side $a, -a \in F$ and so, by Lemma 35, $a \wedge -a \in F$. By S4, then, $b, -b \in F$, i.e. $b \in J$.

3. implies 1. From [6], [26], we know that $\vDash_2^{\mathbb{A}}$ is axiomatisable with MP and RA as its sole rules (cp. the next Section). Then we must prove that, if $F = \uparrow J$, then F contains all the evaluations of the axioms of $\vDash_2^{\mathbb{A}}$ and is closed w.r.t. modus ponens and adjunction. Since $0 \in J$, $P \subseteq F$ and thus every evaluation of every axiom belongs to F. Suppose $a, a \rightarrow b \in F$. Then there are $c, d \in J$ s.t. $c \leq a, d \leq a \rightarrow b$, whence $c + d \leq a - a + b = b$. This means that $b \in F$, since $c + d \in J$. Finally, suppose that $a, b \in F$. Then there are $c, d \in J$ s.t. $c \leq a, d \leq b$, whence $c \wedge d \leq a \wedge b$. This gives us our conclusion. ■

Observe that the equivalence between statements 1. and 2. in Lemma 36 can also be obtained as a corollary to Theorem 31. It is enough to check that congruence filters are closed w.r.t. the rules R1-R6, and that $\vDash_2^{\mathbb{A}}$-filters, as characterised therein, are closed w.r.t. the conditions S1-S4.

4 Hilbert-style systems

Throughout the rest of this paper, we will set aside the general framework of logics from e-ℓ-varieties and focus more closely on the special case of Abelian logics. Whenever an abstract logic is defined semantically, a question naturally arises as to whether such logic admits of a finite axiomatisation by means of a Hilbert-style system. In this section, we try to give comprehensive answers to this question with respect to our Abelian logics.

We start with $\vDash_1^{\mathbb{A}}$ and recall from Example 15 that $\alpha \to \beta$ is, in the present case, a notational variant of $\beta - \alpha$. Consider the following list of axioms and rules:

B $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
C $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$
I $\alpha \to \alpha$
A $((\alpha \to \beta) \to \beta) \to \alpha$
a1 $(\alpha \to \alpha) \to 0$
a2 $(\alpha \to 0) \to -\alpha$
a3 $(\alpha \wedge \beta) \wedge \gamma \to \alpha \wedge (\beta \wedge \gamma)$
a4 $\alpha \wedge \beta \to \beta \wedge \alpha$
a5 $\alpha \to \alpha \wedge \alpha$
a6 $(\alpha \vee \beta) \vee \gamma \to \alpha \vee (\beta \vee \gamma)$
a7 $\alpha \vee \beta \to \beta \vee \alpha$
a8 $\alpha \to \alpha \vee \alpha$
a9 $\alpha \to \alpha \vee (\alpha \wedge \beta)$
a10 $\alpha \to \alpha \wedge (\alpha \vee \beta)$
a11 $\alpha \vee (\beta \wedge \gamma) \to (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
a12 $(\alpha \to \beta) \wedge (\alpha \to \gamma) \to (\alpha \to \beta \wedge \gamma)$
a13 $(\alpha \to \gamma) \wedge (\beta \to \gamma) \to (\alpha \vee \beta \to \gamma)$
M1 $\alpha \to \beta \vdash \alpha \wedge \gamma \to \beta \wedge \gamma$
M2 $\alpha \to \beta \vdash \alpha \vee \gamma \to \beta \vee \gamma$
MP $\alpha, \alpha \to \beta \vdash \beta$
RA $\alpha, \beta \vdash \alpha \wedge \beta$
GR $\alpha, \beta \vdash \alpha \to \beta$

We can define a logic out of this Hilbert style system in the usual way:

Definition 37 *Let* $\Gamma \cup \{\alpha\}$ *be a finite subset of* $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$*. A* 1-proof *of* α *from* Γ *is a finite sequence* $S = \langle \alpha_1, ..., \alpha_n, \alpha \rangle$ *of members of* $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$ *s.t. every member of* S *either (i) is a member of* Γ*; (ii) is a formula in* $\{$*B, C, I, A, a1-13*$\}$*; (iii) is obtained from preceding formulae by means of a rule in* $\{$*M1-2, MP, RA, GR*$\}$*.*

Definition 38 *Let* $\Gamma \cup \{\alpha\}$ *be a finite subset of* $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$*. We define:*

$$\Gamma \vdash_1^{\mathbb{A}} \alpha \textit{ iff there is a } 1\textit{-proof of } \alpha \textit{ from } \Gamma.$$

Remark that the previous list of axioms and rules is highly redundant. For example, it has been observed that both axiom C and axiom I are provable from B, A with the help of modus ponens (MP) [27]. We now establish a few useful syntactic lemmas.

Lemma 39 **L1** $\alpha \to \beta \vdash_1^{\mathbb{A}} \beta \to \alpha$ *(conversion)*

L2 $\alpha \to \beta \vdash_1^{\mathbb{A}} -\beta \to -\alpha$ *(contraposition)*

L3 $\alpha \to \beta, \gamma \to \delta \vdash_1^{\mathbb{A}} (-\alpha \to \gamma) \to (-\beta \to \delta)$

L4 $\alpha \to \beta, \gamma \to \delta \vdash_1^{\mathbb{A}} \alpha \wedge \gamma \to \beta \wedge \delta$ *($\wedge$-monotonicity)*

L5 $\alpha \to \beta, \gamma \to \delta \vdash_1^{\mathbb{A}} \alpha \vee \gamma \to \beta \vee \delta$ *($\vee$-monotonicity)*

Proof. We only provide a proof of L1 and L3. As regards L1, let $\xi = (\alpha \to \beta) \to \beta$.

1.	$\vdash_1^{\mathbb{A}} \beta \to \beta$	I
2.	$\alpha \to \beta \vdash_1^{\mathbb{A}} (\alpha \to \beta) \to (\beta \to \beta)$	1, GR
3.	$\vdash_1^{\mathbb{A}} (\beta \to \xi) \to ((\xi \to \alpha) \to (\beta \to \alpha))$	B
4.	$\vdash_1^{\mathbb{A}} ((\alpha \to \beta) \to (\beta \to \beta)) \to (\beta \to \xi)$	C
5.	$\alpha \to \beta \vdash_1^{\mathbb{A}} \beta \to \xi$	2, 4, MP
6.	$\alpha \to \beta \vdash_1^{\mathbb{A}} (\xi \to \alpha) \to (\beta \to \alpha)$	3, 5, MP
7.	$\vdash_1^{\mathbb{A}} \xi \to \alpha$	A
8.	$\alpha \to \beta \vdash_1^{\mathbb{A}} \beta \to \alpha$	6, 7, MP

As regards L3,

1.	$\alpha \to \beta \vdash_1^{\mathbb{A}} -\beta \to -\alpha$	L2
2.	$\alpha \to \beta \vdash_1^{\mathbb{A}} (-\alpha \to \gamma) \to (-\beta \to \gamma)$	1, B, MP
3.	$\gamma \to \delta \vdash_1^{\mathbb{A}} (-\beta \to \gamma) \to (-\beta \to \delta)$	B, C, MP
4.	$\alpha \to \beta, \gamma \to \delta \vdash_1^{\mathbb{A}} (-\alpha \to \gamma) \to (-\beta \to \delta)$	2, 3, B, MP

■

A replacement theorem is available for $\vdash_1^{\mathbb{A}}$.

Theorem 40 $\alpha \to \beta \vdash_1^{\mathbb{A}} \gamma \to \gamma\,[\alpha/\beta]$

Proof. Induction on the construction of γ. Base. We distinguish three cases: (i) $\gamma = p = \alpha$. Then of course $p \to \beta \vdash_1^{\mathbb{A}} p \to \beta$. (ii) $\gamma = p \neq \alpha$. Our claim follows from the fact that $\vdash_1^{\mathbb{A}} p \to p$. (iii) $\gamma = 0$. This case is disposed of similarly.

Inductive step. For the cases of $+, \wedge, \vee$, use L3, L4 and L5 respectively. Finally, let $\gamma = -\delta$. Then:

1.	$\alpha \to \beta \vdash_1^{\mathbb{A}} \delta \to \delta\,[\alpha/\beta]$	IH
2.	$\alpha \to \beta \vdash_1^{\mathbb{A}} \delta\,[\alpha/\beta] \to \delta$	1, L1
3.	$\alpha \to \beta \vdash_1^{\mathbb{A}} -\delta \to -\delta\,[\alpha/\beta]$	2, L2

■

The preceding theorem illustrates a typical use of conversion in $\vdash_1^{\mathbb{A}}$-derivations. Hereafter, we will sometimes feel free to use our lemma L1 and Theorem 40 without any special mention.

In the rest of our paper we will make extensive use of the following abbreviations:

Notation 41 *For* $\alpha \in \mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$*, we define:* $\alpha^{+} = \alpha \vee 0$*;* $\alpha^{-} = \alpha \wedge 0$*;* $|\alpha| = \alpha \vee -\alpha$*. Moreover we define:*

$$\begin{aligned} 0\alpha &= 0; \\ 1\alpha &= \alpha; \\ (k+1)\,\alpha &= k\alpha + \alpha. \end{aligned}$$

We now list some more useful syntactic lemmas, together with some hints as to how they are to be proved. Where no hint is given, the reader will find a more detailed proof in Apprendix A.

Lemma 42 **L6** $\vdash_1^{\mathbb{A}} \alpha^{+-} \to 0$ *(a10)*

L7 $\vdash_1^{\mathbb{A}} (\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$ *(B, C)*

L8 $\vdash_1^{\mathbb{A}} ((\gamma \to \alpha) \to (\gamma \to \beta)) \to (\alpha \to \beta)$ *(L7)*

L9 $\vdash_1^{\mathbb{A}} 0 \to (\alpha \to \alpha)$ *(a1)*

L10a $\vdash^{\mathbb{A}}_{1} \alpha \to (0 \to \alpha)$ *(C, L9)*

L10b $\vdash^{\mathbb{A}}_{1} (0 \to \alpha) \to \alpha$ *(L10a)*

L11 $\vdash^{\mathbb{A}}_{1} 0$ *(I, a1)*

L12 $\vdash^{\mathbb{A}}_{1} (\alpha \to \beta) \to ((\beta \to \alpha) \to 0)$ *(B, a1)*

L13 $\vdash^{\mathbb{A}}_{1} (\alpha^{+} \to \beta) \to (\alpha \to \beta) \wedge \beta$ *(a13, L10a)*

L14 $\vdash^{\mathbb{A}}_{1} \alpha \to ((\alpha \to \beta) \to \beta)$ *(B, C, I)*

L15a $\vdash^{\mathbb{A}}_{1} \alpha \wedge \beta \to \alpha \wedge (0 \to \beta)$ *(L10a, M1)*

L15b $\vdash^{\mathbb{A}}_{1} \alpha \wedge (0 \to \beta) \to \alpha \wedge \beta$ *(L15a)*

L16 $\vdash^{\mathbb{A}}_{1} \alpha \wedge \beta \to \alpha \wedge ((\beta \to \gamma) \to \gamma)$ *(L14, M1)*

L17 $\vdash^{\mathbb{A}}_{1} \alpha \vee \beta \to \alpha \vee (0 \to \beta)$ *(L10a, M2)*

L18 $\vdash^{\mathbb{A}}_{1} (\alpha \to \beta \vee (0 \to \gamma)) \to (\alpha \to \beta \vee \gamma)$ *(L7, L17)*

L19 $\vdash^{\mathbb{A}}_{1} (\gamma \wedge (\alpha \to \beta) \to \beta) \to \alpha \vee (\gamma \to \beta)$

L20 $\vdash^{\mathbb{A}}_{1} \alpha^{-} \to (\alpha^{+} \to \alpha)$

L21 $\vdash^{\mathbb{A}}_{1} \Big(\alpha \to \Big((\alpha \to \beta)^{-} \to \beta\Big)\Big) \to (\alpha \to \beta)^{+}$

L22 $\vdash^{\mathbb{A}}_{1} \alpha^{-} \to \alpha \wedge (\beta \to \beta)$ *(L9, M1)*

L23 $\vdash^{\mathbb{A}}_{1} (0 \to \alpha^{-}) \to \alpha \wedge (\beta \to \beta)$ *(L10, L22)*

L24 $\vdash^{\mathbb{A}}_{1} (\alpha \to \beta)^{-} \to (\beta \vee \alpha \to \beta)$

L25 $\vdash^{\mathbb{A}}_{1} ((\alpha \to \beta) \wedge \gamma \to \beta) \to \alpha \vee (\gamma \to \beta)$ *(L19, a4)*

L26 $\vdash^{\mathbb{A}}_{1} ((\beta \to \alpha) \to 0)^{+} \to (\alpha \to \beta)^{+}$ *(L12, M2)*

L27 $\vdash^{\mathbb{A}}_{1} (\alpha \to \beta)^{+} \to (\alpha \to \alpha \vee \beta)$

Equipped with this lemma, we can now prove that our axiom system actually axiomatises the logic $\vDash^{\mathbb{A}}_{1}$.

Theorem 43 *Let $\Gamma \cup \{\alpha\}$ be a finite subset of $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$. Then $\Gamma \vdash^{\mathbb{A}}_{1} \alpha$ iff $\Gamma \vDash^{\mathbb{A}}_{1} \alpha$.*

Proof. The left-to-right direction is shown by induction on the length of the 1-proof of α from Γ. For the converse, given the results in [18], it suffices to prove that the translations into $\mathcal{L}$ of all the postulates of the Hilbert system $\mathcal{BAL}$ therein defined are derivable in our system. This is obvious in all cases except for the axioms P and O and for the rules PI and MI. We provide below streamlined versions of these derivations.

(Axiom P)

$$\begin{array}{lll}
1. & \vdash_1^{\mathbb{A}} (\alpha^{++} \to \alpha^{+}) \to (\alpha^{+} \to \alpha^{+}) \wedge \alpha^{+} & \text{L13} \\
2. & \vdash_1^{\mathbb{A}} (\alpha^{++} \to \alpha^{+}) \to \alpha^{+-} & \text{1, a1, L9} \\
3. & \vdash_1^{\mathbb{A}} (\alpha^{++} \to \alpha^{+}) \to 0 & \text{2, L6} \\
4. & \vdash_1^{\mathbb{A}} \alpha^{++} \to \alpha^{+} & \text{3, L1, L11}
\end{array}$$

(Axiom O)

$$\begin{array}{lll}
1. & \vdash_1^{\mathbb{A}} ((\beta \to \alpha) \to 0)^{+} \to \Big((\beta \to \alpha) \to (\beta \to \alpha)^{+}\Big) & \text{L27} \\
2. & \vdash_1^{\mathbb{A}} ((\beta \to \alpha) \to 0)^{+} \to (\alpha \to \beta)^{+} & \text{L26} \\
3. & \vdash_1^{\mathbb{A}} 1 \to 2 & \text{1, 2, GR} \\
4. & \vdash_1^{\mathbb{A}} (\alpha \to \beta)^{+} \to \Big((\beta \to \alpha) \to (\beta \to \alpha)^{+}\Big) & \text{3, L8} \\
5. & \vdash_1^{\mathbb{A}} (\beta \to \alpha) \to \Big((\alpha \to \beta)^{+} \to (\beta \to \alpha)^{+}\Big) & \text{4, C} \\
6. & \vdash_1^{\mathbb{A}} \Big((\alpha \to \beta)^{+} \to (\beta \to \alpha)^{+}\Big) \to (\beta \to \alpha) & \text{5, L1}
\end{array}$$

(Rule PI)

$$\begin{array}{lll}
1. & \alpha \vdash_1^{\mathbb{A}} 0 & \text{L11} \\
2. & \alpha \vdash_1^{\mathbb{A}} 0^{+} & \text{1, a8} \\
3. & \alpha \vdash_1^{\mathbb{A}} 0 \to \alpha & \text{1, GR} \\
4. & \alpha \vdash_1^{\mathbb{A}} 0^{+} \to \alpha^{+} & \text{3, M2} \\
5. & \alpha \vdash_1^{\mathbb{A}} \alpha^{+} & \text{2, 4, MP}
\end{array}$$

(Rule MI)

$$\begin{array}{lll}
1. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} \beta \vee \alpha \to \alpha & \text{L1, L27} \\
2. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} (\alpha \vee \beta \to \alpha) \to 0 & \text{1, a7, L11, GR} \\
3. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} (\beta \to \alpha)^{-} \to 0 & \text{2, B, L24} \\
4. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} 0 \to (\beta \to \alpha)^{-} & \text{3, L1} \\
5. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} (\beta \to \beta) \wedge (\beta \to \alpha) & \text{4, L23, a4} \\
6. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} \beta \to \beta \wedge \alpha & \text{5, a12} \\
7. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} (\beta \wedge \alpha)^{+} \to \beta^{+} & \text{6, L1, M2} \\
8. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} \big(\beta^{+} \wedge \alpha^{+}\big) \to \beta^{+} & \text{7, a11} \\
9. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} \big(\big(0 \to \beta^{+}\big) \wedge \alpha^{+}\big) \to \beta^{+} & \text{8, L10} \\
10. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} \Big(\Big(\big(0 \to \beta^{+}\big)^{+} \wedge \alpha^{+}\Big) \to \beta^{+}\Big) \to \big(\alpha^{+} \to \beta^{+}\big)^{+} & \text{L25} \\
11. & (\alpha \to \beta)^{+} \vdash_1^{\mathbb{A}} \big(\alpha^{+} \to \beta^{+}\big)^{+} & \text{9, 10, MP}
\end{array}$$

■

The problem of axiomatising $\models_2^{\mathbb{A}}$ has already been solved by Meyer, Slaney and Casari. Consider the following formulae:

$$\begin{array}{ll}
\text{(a14)} & \alpha \wedge \beta \to \alpha,\ \alpha \wedge \beta \to \beta \\
\text{(a15)} & \alpha \to \alpha \vee \beta,\ \beta \to \alpha \vee \beta
\end{array}$$

Definition 44 *Let* $\Gamma \cup \{\alpha\}$ *be a finite subset of* $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$*. A* 2-proof *of* α *from* Γ *is a finite sequence* $S = \langle \alpha_1, ..., \alpha_n, \alpha \rangle$ *of members of* $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$ *s.t. every member of* S *either (i) is a member of* Γ*; (ii) is a formula in* $\{B, C, I, A, a1, a2, a11\text{-}a15\}$*; (iii) is obtained from preceding formulae by means of a rule in* $\{MP, RA\}$*.*

Definition 45 *Let* $\Gamma \cup \{\alpha\}$ *be a finite subset of* $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$*. We define:*

$$\Gamma \vdash_2^{\mathbb{A}} \alpha \text{ iff there is a 2-proof of } \alpha \text{ from } \Gamma.$$

Meyer and Slaney [26] confine themselves to a weak completeness theorem for the logic so axiomatised; Casari [6], on the other hand, proves a *strong* completeness theorem but provides a different axiomatisation. Various results scattered in [26], [6], [29], [30], [20], however, imply that

Theorem 46 *Let* $\Gamma \cup \{\alpha\}$ *be a finite subset of* $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$*. Then* $\Gamma \vdash_2^{\mathbb{A}} \alpha$ *iff* $\Gamma \vDash_2^{\mathbb{A}} \alpha$*.*

We leave the axiomatisation of $\vDash_3^{\mathbb{A}}$ as an open problem for the interested reader.

5 Deduction theorems

Deduction-detachment theorems for Abelian logics appeared both in [26], section IX, and in [18]. Meyer and Slaney provide what in fact is a deduction theorem for $\vDash_2^{\mathbb{A}}$, but omit its proof; they also hint at a deduction theorem for a logic which looks sufficiently like $\vDash_3^{\mathbb{A}}$, but express their results in the jargon of relevant "theories" (which are not the standard theories of AAL in that they need not be closed w.r.t. the rules of the logic under consideration), a circumstance which renders a comparison difficult. Galli et al., on the other hand, prove a deduction theorem for $\vDash_1^{\mathbb{A}}$, but it is not so clear from the statement of the result that we have to do with a *local* deduction-detachment theorem in the sense of AAL.

The aim of this section is repairing these flaws in the literature, proving in detail three deduction theorems (two *local* DDTs and a *graded* DDT) for the three logics under consideration. We start with an easy "graded" deduction theorem which holds even in the general case of a logic $\vDash_3^{\mathbb{V}}$ from an arbitrary e-ℓ-variety with implication (cp. [11]).

Theorem 47 *Let* $\mathbb{V}$ *be a* e-ℓ*-variety of type* $\mathcal{L}$ *with implication* $\rightarrow$*. Then* $\Gamma \vDash_3^{\mathbb{V}} \beta$ *iff* $e \vDash_3^{\mathbb{V}} \bigwedge \{\gamma : \gamma \in \Gamma\} \rightarrow \beta$*.*

Proof. In fact, $\Gamma \vDash_3^{\mathbb{V}} \beta$ holds iff for any $\overrightarrow{a} \in A \in \mathbb{V}$, $\bigwedge^{\mathbf{A}} \{\gamma^{\mathbf{A}} (\overrightarrow{a}) : \gamma \in \Gamma\} \leq^{\mathbf{A}} \beta^{\mathbf{A}} (\overrightarrow{a})$, which in turn holds iff for any $\overrightarrow{a} \in A \in \mathbb{V}$, $e^{\mathbf{A}} \leq^{\mathbf{A}} \left(\bigwedge^{\mathbf{A}} \{\gamma^{\mathbf{A}} (\overrightarrow{a}) : \gamma \in \Gamma\}\right) \rightarrow^{\mathbf{A}} \beta^{\mathbf{A}} (\overrightarrow{a})$, i.e. iff $e \vDash_3^{\mathbb{V}} \bigwedge \{\gamma : \gamma \in \Gamma\} \rightarrow \beta$. ■

We now come to $\vDash_1^{\mathbb{A}}$.

Theorem 48 *Let* $\Gamma \cup \{\alpha, \beta\}$ *be a finite subset of* $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$*. Then*

$$\Gamma, \alpha \vDash_1^{\mathbb{A}} \beta \text{ iff there exists } n \in N \text{ s.t. } \Gamma \vDash_1^{\mathbb{A}} n\,|\alpha| \rightarrow |\beta|\,.$$

Proof. By Proposition 4.43 in [21], $\vDash_1^{\mathbb{A}}$ has a local DDT relative to the set

$$\Phi = \{\{i\,|\alpha| \to |\beta|\} : i \in N\}$$

iff for every $\langle 2,2,2,1,0\rangle$-algebra $\mathbf{A}$, every $\vDash_1^{\mathbb{A}}$-filter on $\mathbf{A}$, and every $a, b \in A$,

$$b \in Fi_{\vDash_1}^{\mathbf{A}}(F, a) \text{ iff } \Delta^{\mathbf{A}}(a, b) \subseteq F \text{ for some } \Delta \in \Phi.$$

Since $\vDash_1^{\mathbb{A}}$ is a regularly algebraisable logic, by Theorem 5.2.7 in [7] we can replace "$\langle 2,2,2,1,0\rangle$-algebra" by "Abelian ℓ-group" in the biconditional above. Also, by Corollary 28, what we must prove is that for every convex ℓ-subgroup F of $\mathbf{A}$ and every $a, b \in A$,

$$b \in (F \cup \{a\}] \text{ iff there exists } n \in N \text{ s.t. } n\,|a| \to |b| \in F,$$

where $(F \cup \{a\}]$ denotes the convex ℓ-subgroup generated by F, a in $\mathbf{A}$. However, this much holds true as

$$\begin{aligned} b \in (F \cup \{a\}] \quad & \text{iff } |b| \leq n\,|a| + \sum_{i \leq m} |c_i| \text{ for some } c_1, ..., c_m \in F \\ & \text{iff } -n\,|a| + |b| \leq \sum_{i \leq m} |c_i| \text{ for some } c_1, ..., c_m \in F \\ & \text{iff } n\,|a| \to |b| \leq \sum_{i \leq m} |c_i| \text{ for some } c_1, ..., c_m \in F \\ & \text{iff } n\,|a| \to |b| \in F. \end{aligned}$$

■

Observe that the previous theorem implies the CEP for Abelian ℓ-groups, a result which is somewhat surprisingly not listed among the properties of $\mathbb{A}$ in the Mathematical Structures Homepage, although Abelian ℓ-groups certainly enjoy the property when formulated in the similarity type of residuated lattices, as a consequence of results in [15] (remark, however, that term equivalence does not preserve subalgebras).

The following local DDT for $\vDash_2^{\mathbb{A}}$ is stated, but not proved, in [26] (while a more general result is proved in [17], Theorem 4.9). We think it may be instructive to provide a detailed proof of it. To achieve this goal, we still need some further syntactic lemmas, consequences of Theorem 46:

L28 $\vdash_2^{\mathbb{A}} (\alpha \to (\beta \to \gamma)) \to ((\alpha + \beta) \to \gamma)$

L29 $\vdash_2^{\mathbb{A}} \left(\alpha^- + \beta^-\right) \to (\alpha + \beta)^-$

L30 $\alpha \vdash_2^{\mathbb{A}} n\alpha$ for every $n \geq 0$.

Theorem 49 *Let* $\Gamma \cup \{\alpha, \beta\}$ *be a finite subset of* $\mathtt{Fm}(\mathcal{L}^{\mathbb{A}})$. *Then*

$$\Gamma, \alpha \vDash_2^{\mathbb{A}} \beta \textit{ iff there exists } n \in N \textit{ s.t. } \Gamma \vDash_2^{\mathbb{A}} n\alpha \supset \beta.$$

Proof. By Theorem 46 we can prove the theorem by syntactical means. We begin with the left-to-right direction, which is proved by induction on the 2-proof of β from Γ, α.

(i) β is an axiom. Choose $n = 0$; by a5, L10 (which are easily checked to hold also for $\vdash_2^{\mathbb{A}}$), $\Gamma \vdash_2^{\mathbb{A}} 0^- \to \beta$, i.e. $\Gamma \vdash_2^{\mathbb{A}} 0 \supset \beta$.

(ii) $\beta = \alpha$. Choose $n = 1$; by a14, $\Gamma \vdash_2^{\mathbb{A}} \beta^- \to \beta$, i.e. $\Gamma \vdash_2^{\mathbb{A}} \beta \supset \beta$.

(iii) $\beta \in \Gamma$. This means that $\Gamma, \beta \vdash_2^{\mathbb{A}} \beta$. Choose $n = 0$; then by a5, L10 $\Gamma, \beta \vdash_2^{\mathbb{A}} 0 \supset \beta$.

(iv) β is obtained by MP. Then for some formula γ we have both $\Gamma, \alpha \vdash_2^{\mathbb{A}} \gamma$ and $\Gamma, \alpha \vdash_2^{\mathbb{A}} \gamma \to \beta$. By inductive hypothesis, there exist natural numbers n_1, n_2 s.t. $\Gamma \vdash_2^{\mathbb{A}} n_1\alpha \supset \gamma$ and $\Gamma \vdash_2^{\mathbb{A}} n_2\alpha \supset (\gamma \to \beta)$, i.e. $\Gamma \vdash_2^{\mathbb{A}} (n_1\alpha)^- \to \gamma$ and $\Gamma \vdash_2^{\mathbb{A}} (n_2\alpha)^- \to (\gamma \to \beta)$, whence by axioms C and B $\Gamma \vdash_2^{\mathbb{A}} \gamma \to \left((n_2\alpha)^- \to \beta\right)$ and $\Gamma \vdash_2^{\mathbb{A}} (n_1\alpha)^- \to \left((n_2\alpha)^- \to \beta\right)$. By L28, $\Gamma \vdash_2^{\mathbb{A}} \left((n_1\alpha)^- + (n_2\alpha)^-\right) \to \beta$, whence, setting $n = n_1 + n_2$, we obtain by L29 $\Gamma \vdash_2^{\mathbb{A}} n\alpha \supset \beta$.

(v) $\beta = \gamma \wedge \delta$ is obtained by RA. Then for some formulae γ, δ we have both $\Gamma, \alpha \vdash_2^{\mathbb{A}} \gamma$ and $\Gamma, \alpha \vdash_2^{\mathbb{A}} \delta$. By inductive hypothesis, there exist natural numbers n_1, n_2 s.t. $\Gamma \vdash_2^{\mathbb{A}} (n_1\alpha)^- \to \gamma$ and $\Gamma \vdash_2^{\mathbb{A}} (n_2\alpha)^- \to \delta$. Setting $n = \max(n_1 + n_2)$, by a14 $\Gamma \vdash_2^{\mathbb{A}} (n\alpha)^- \to \gamma$ and $\Gamma \vdash_2^{\mathbb{A}} (n\alpha)^- \to \delta$, whence by a12 $\Gamma \vdash_2^{\mathbb{A}} (n\alpha)^- \to \gamma \wedge \delta$, i.e. $\Gamma \vdash_2^{\mathbb{A}} n\alpha \supset \gamma \wedge \delta$.

Right to left:

1.	$\Gamma \vdash_2^{\mathbb{A}} n\alpha \supset \beta$	hyp.
2.	$\Gamma, \alpha \vdash_2^{\mathbb{A}} n\alpha$	L30
3.	$\Gamma, \alpha \vdash_2^{\mathbb{A}} 0$	I, a1
4.	$\Gamma, \alpha \vdash_2^{\mathbb{A}} (n\alpha)^-$	2, 3, RA
5.	$\Gamma, \alpha \vdash_2^{\mathbb{A}} \beta$	1, 4, MP

■

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6 Appendix A: Proof of Lemma 42

L19

$$\begin{array}{rll}
1. & \vdash^{\mathbb{A}}_{1} (\alpha \to \beta) \wedge \gamma \to (\alpha \to \beta) \wedge ((\gamma \to \beta) \to \beta) & \text{L16} \\
2. & \vdash^{\mathbb{A}}_{1} (\alpha \to \beta) \wedge ((\gamma \to \beta) \to \beta) \to (\alpha \vee (\gamma \to \beta) \to \beta) & \text{a13} \\
3. & \vdash^{\mathbb{A}}_{1} (\alpha \to \beta) \wedge \gamma \to (\alpha \vee (\gamma \to \beta) \to \beta) & \text{1, 2, B} \\
4. & \vdash^{\mathbb{A}}_{1} \gamma \wedge (\alpha \to \beta) \to ((\gamma \to \beta) \vee \alpha \to \beta) & \text{3, a4} \\
5. & \vdash^{\mathbb{A}}_{1} (\gamma \to \beta) \vee \alpha \to (\gamma \wedge (\alpha \to \beta) \to \beta) & \text{4, C} \\
6. & \vdash^{\mathbb{A}}_{1} (\gamma \wedge (\alpha \to \beta) \to \beta) \to \alpha \vee (\gamma \to \beta) & \text{5, L1, a7}
\end{array}$$

L20

$$\begin{array}{lll}
1. & \vdash^{\mathbb{A}}_{1} \Big((0 \to \alpha)^{-} \to \alpha\Big) \to (0 \to \alpha)^{+} & \text{L19} \\
2. & \vdash^{\mathbb{A}}_{1} \Big((0 \to \alpha)^{-} \to \alpha\Big) \to \alpha^{+} & \text{1, L10} \\
3. & \vdash^{\mathbb{A}}_{1} \alpha^{+} \to \Big((0 \to \alpha)^{-} \to \alpha\Big) & \text{2, L1} \\
4. & \vdash^{\mathbb{A}}_{1} (0 \to \alpha)^{-} \to (\alpha^{+} \to \alpha) & \text{3, C} \\
5. & \vdash^{\mathbb{A}}_{1} \alpha^{-} \to (\alpha^{+} \to \alpha) & \text{4, L10}
\end{array}$$

L21

$$\begin{array}{lll}
1. & \vdash^{\mathbb{A}}_{1} (\alpha \to \beta)^{+} \to \Big((\alpha \to \beta)^{-} \to (\alpha \to \beta)\Big) & \text{L20, C} \\
2. & \vdash^{\mathbb{A}}_{1} \Big((\alpha \to \beta)^{-} \to (\alpha \to \beta)\Big) \to (\alpha \to \beta)^{+} & \text{1, L1} \\
3. & \vdash^{\mathbb{A}}_{1} \Big(\alpha \to \Big((\alpha \to \beta)^{-} \to \beta\Big)\Big) \to (\alpha \to \beta)^{+} & \text{2, C}
\end{array}$$

L24

$$\begin{array}{lll}
1. & \vdash^{\mathbb{A}}_{1} (\alpha \to \beta)^{-} \to (\alpha \to \beta) \wedge (\beta \to \beta) & \text{L23} \\
2. & \vdash^{\mathbb{A}}_{1} (\alpha \to \beta) \wedge (\beta \to \beta) \to (\beta \vee \alpha \to \beta) & \text{a13, a7} \\
3. & \vdash^{\mathbb{A}}_{1} (\alpha \to \beta)^{-} \to (\beta \vee \alpha \to \beta) & \text{1, 2, B}
\end{array}$$

L27

$$\begin{array}{lll}
1. & \vdash^{\mathbb{A}}_{1} \Big(\alpha \to \Big((\alpha \to \beta)^{-} \to \beta\Big)\Big) \to (\alpha \to \alpha \vee (0 \to \beta)) & \text{L19, L7} \\
2. & \vdash^{\mathbb{A}}_{1} \Big(\alpha \to \Big((\alpha \to \beta)^{-} \to \beta\Big)\Big) \to (\alpha \to \beta)^{+} & \text{L21} \\
3. & \vdash^{\mathbb{A}}_{1} 1 \to 2 & \text{1, 2, GR} \\
4. & \vdash^{\mathbb{A}}_{1} (\alpha \to \alpha \vee (0 \to \beta)) \to (\alpha \to \beta)^{+} & \text{3, L8} \\
5. & \vdash^{\mathbb{A}}_{1} (\alpha \to \alpha \vee (0 \to \beta)) \to (\alpha \to \alpha \vee \beta) & \text{L18} \\
6. & \vdash^{\mathbb{A}}_{1} 4 \to 5 & \text{4, 5, GR} \\
7. & \vdash^{\mathbb{A}}_{1} (\alpha \to \beta)^{+} \to (\alpha \to \alpha \vee \beta) & \text{7, L8}
\end{array}$$